Poynting Vector

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$$

It represents the instantaneous flow of electromagnetic power per unit area (unit W/m^2). The direction of power flow is normal to the plane containing $\boldsymbol{E}$ and $\boldsymbol{H}$, in the sense of $\boldsymbol{E} \times \boldsymbol{H}$ — for a plane wave, the direction of propagation.

For sinusoidally varying fields the time-averaged power per unit area (the intensity of the wave) is

$$\langle \mathbf{S} \rangle = \frac{1}{2} E_0 H_0 = \frac{{E_0}^2}{2Z} = \frac{{E_{rms}}^2}{Z}$$

where E_0 and H_0 are the amplitudes and $Z = E_0/H_0 = \sqrt{\frac{\mu}{\epsilon}}$ is the wave impedance (377 Ω in free space). A plane wave carries the energy density $u = \frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2 = \epsilon E^2$ along with it at the wave speed v, so that $S = uv$.

Poynting Theorem

$$\boldsymbol{\nabla} \cdot \boldsymbol{S} + \mathbf{J} \cdot \boldsymbol{E} + \mathbf{H} \cdot \frac{\boldsymbol{\partial B}}{\boldsymbol{\partial} t} + \mathbf{E} \cdot \frac{\boldsymbol{\partial D}}{\boldsymbol{\partial} t} = 0$$

This is Poynting's theorem — the law of conservation of energy for the electromagnetic field. (It follows from the vector identity $\boldsymbol{\nabla} \cdot (\boldsymbol{E} \times \boldsymbol{H}) = \mathbf{H} \cdot (\boldsymbol{\nabla} \times \mathbf{E}) - \mathbf{E} \cdot (\boldsymbol{\nabla} \times \mathbf{H})$ and the two curl equations of Maxwell; the derivation is not required.)

For time varying fields in a linear, homogeneous and isotropic medium

$$\mathbf{B} = \mu \mathbf{H} \qquad \mathbf{D} = \epsilon \mathbf{E}$$

$$\mathbf{H} \cdot \frac{\partial \boldsymbol{B}}{\partial t} = \frac{1}{2} \frac{\partial (\boldsymbol{B} \cdot \mathbf{H})}{\partial t} = \frac{1}{2} \frac{\partial (\mu \mathbf{H}^2)}{\partial t}$$

$$\mathbf{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} = \frac{1}{2} \frac{\partial (\boldsymbol{E} \cdot \mathbf{D})}{\partial t} = \frac{1}{2} \frac{\partial (\epsilon \mathbf{E}^2)}{\partial t}$$

$$\boldsymbol{\nabla \cdot S} + \mathbf{J} \cdot \boldsymbol{E} + \frac{\partial}{\partial t}\left(\frac{1}{2}\mu \mathbf{H}^2\right) + \frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon \mathbf{E}^2\right) = 0$$

Writing the magnetic and electric energy densities as $w_m = \frac{1}{2}\mu \mathbf{H}^2$ and $w_e = \frac{1}{2}\epsilon \mathbf{E}^2$:

$$\boldsymbol{\nabla \cdot S} + \mathbf{J} \cdot \boldsymbol{E} + \frac{\partial}{\partial t}(w_m) + \frac{\partial}{\partial t}(w_e) = 0$$

$$\int_{\mathbf{v}} \boldsymbol{\nabla \cdot S}\, d\boldsymbol{v} + \int_{\mathbf{v}} \mathbf{J} \cdot \boldsymbol{E}\, d\boldsymbol{v} + \int_{\mathbf{v}} \frac{\partial}{\partial t}(w_m)\, d\boldsymbol{v} + \int_{\mathbf{v}} \frac{\partial}{\partial t}(w_e)\, d\boldsymbol{v} = 0$$

$$\oint_{\mathbf{s}} \mathbf{S} \cdot \boldsymbol{dS} + \int_{\mathbf{v}} \mathbf{J} \cdot \boldsymbol{E}\, d\boldsymbol{v} + \int_{\mathbf{v}} \frac{\partial}{\partial t}(w_m)\, d\boldsymbol{v} + \int_{\mathbf{v}} \frac{\partial}{\partial t}(w_e)\, d\boldsymbol{v} = 0$$

$$-\oint_{\mathbf{s}} \mathbf{S} \cdot \boldsymbol{dS} = \int_{\mathbf{v}} \mathbf{J} \cdot \boldsymbol{E}\, d\boldsymbol{v} + \frac{d}{dt}\left(\int_{\mathbf{v}} (w_m)\, d\boldsymbol{v} + \int_{\mathbf{v}} (w_e)\, d\boldsymbol{v}\right)$$

Net power flow into the volume **v =** rate of Joule heating (power dissipated as heat) + rate of increase of the energy stored in the electric and magnetic fields.

For static fields

$$-\oint_{\mathbf{s}} \mathbf{S} \cdot \boldsymbol{dS} = \int_{\mathbf{v}} \mathbf{J} \cdot \boldsymbol{E}\, d\boldsymbol{v}$$

Poynting vector inside a coaxial cable

The Poynting vector within the dielectric insulator of a coaxial cable is nearly parallel to the axis: the electrical energy flows through the dielectric between the conductors, not through the metal. No energy flows in ideal conductors themselves, since the electric field inside them is zero (in a real cable a small inward component of S at the conductor surfaces supplies the Joule heating of the wires). No energy flows outside the cable either, since there the magnetic fields of the inner and outer conductors cancel.

Tutorial Questions

1. Calculate the value of Poynting vector at the surface of Sun if the power radiated by the Sun is 3.8×10^{26}W and its radius is 7×10^8 m. Ans: $S = P/(4\pi R^2) \approx 6.2 \times 10^7$ W/m^2.
2. If the earth receives 2 cal min^{-1} cm^{-2} solar energy, what would be the amplitudes of electric and magnetic fields of radiation. Ans: 2 cal min^{-1} cm^{-2} = 1.4×10^3 W/m^2; with $\langle S \rangle = E_0^2/2Z_0$: $E_0 \approx 1.0 \times 10^3$ V/m, $H_0 = E_0/Z_0 \approx 2.7$ A/m ($B_0 \approx 3.4$ μT).
3. If the magnitude of H in a plane wave is 1 A/m, find the magnitude of E for plane wave in free space. Ans: $E = Z_0 H = 377$ V/m.

4. Considering that all the energy from a 1000W lamp is radiated uniformly, calculate the average value of the intensity of electric field of radiation at a distance of 2m from the lamp. Ans: $\langle S \rangle = 1000/(4\pi \times 2^2) \approx 19.9$ W/m^2; $E_0 = \sqrt{(2Z_0 \langle S \rangle)} \approx 122$ V/m (rms value 87 V/m).
5. If the relative permittivity of distilled water is 81, calculate refractive index and velocity of light in it. Ans: $n = \sqrt{\epsilon_r} = 9$, $v = c/n \approx 3.3 \times 10^7$ m/s — using the static ϵ_r; at optical frequencies water has n = 1.33.

Applications

- Sunlight: the solar constant, $\langle S \rangle \approx 1.36$ kW/m^2 above the atmosphere (about 1 kW/m^2 at the ground), is the figure that sizes every solar panel and solar water heater; it corresponds to a field amplitude of about 1 kV/m.
- Energy travels in the field, not in the wire: for a wire carrying a current, $\mathbf{E}$ is along the wire and $\mathbf{H}$ circles it, so $\mathbf{S}$ points radially inward at the surface, and the total inflow equals the Joule heating I^2R. The energy from a battery to a lamp travels through the fields around the wires — which is also why the coaxial cable above carries its power in the dielectric.
- Radiation pressure: an absorbed wave delivers momentum at the rate S/c per unit area (2S/c if reflected). Sunlight exerts only ~5 μPa, yet that is enough to drive solar sails and to shape the tails of comets; a focused laser beam exerts enough to hold a living cell in optical tweezers.
- Safety and engineering limits are power densities: public exposure limits for radio-frequency fields are of the order of 10 W/m^2 at mobile-phone frequencies; a laser's eye-safety class is decided by its power divided by the beam area, i.e. its Poynting flux; a microwave oven delivers about 1 kW into its cavity.
- Antennas and links: the power an antenna collects is $\langle S \rangle$ times its effective area, and the inverse-square fall-off $\langle S \rangle = P/(4\pi r^2)$ is the link budget of every satellite, broadcast and mobile link — tutorial questions 1 and 4 are exactly this calculation.

Explore further

- The Feynman Lectures on Physics, Vol. II, Chapter 27 "Field Energy and Field Momentum" (free online) — the energy density and Poynting vector, with the charging capacitor, the current-carrying wire and the magnet-plus-charge paradox as examples.
- Cochran & Heinrich, Applications of Maxwell's Equations §8.2 "Poynting's Theorem" (LibreTexts) — the full derivation for those who want it, applied to dipole radiation.

- Veritasium: “The Big Misconception About Electricity” (video, 14 min) — a provocative demonstration that the energy in a circuit is carried by the fields, i.e. by the Poynting vector, not by the electrons in the wire.