Wave impedance and skin depth

General wave equation

$$\boldsymbol{\nabla^2 u} - \frac{1}{v^2}\frac{\partial^2 \mathbf{u}}{\partial \mathrm{t}^2} = 0$$

$$v = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{\sqrt{\mu_r \epsilon_r}}$$

The plane wave solutions are

$$\boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{E_0} \boldsymbol{e^{i(k \cdot r - \omega t)}}$$

$$\boldsymbol{H}(\boldsymbol{r}, t) = \boldsymbol{H_0} \boldsymbol{e^{i(k \cdot r - \omega t)}}$$

The following Maxwell equations give

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\mu \frac{\partial \mathbf{H}}{\partial \mathrm{t}} \qquad \boldsymbol{\nabla} \times \boldsymbol{H} = \epsilon \frac{\partial \mathbf{E}}{\partial \mathrm{t}}$$

$$\mathbf{k} \times \boldsymbol{E} = \mu\omega \mathbf{H} \qquad \mathbf{k} \times \boldsymbol{H} = -\epsilon\omega \mathbf{E}$$

The first of these gives, since $\mathbf{k}$, $\mathbf{E}$ and $\mathbf{H}$ are mutually perpendicular (the wave is transverse) so that $|\mathbf{k} \times \mathbf{E}| = kE$:

$$\frac{\omega\mu}{\mathrm{k}} = \left|\frac{E}{\mathrm{H}}\right|$$

Since $v = \frac{\omega}{k}$

Wave impedance $Z = \left|\frac{\boldsymbol{E}}{\boldsymbol{H}}\right| = v\mu$ (unit: ohm, since V/m ÷ A/m = Ω; also called the intrinsic impedance of the medium)

(*In acoustics the expression of wave impedance is given by* $Z = \frac{\boldsymbol{P}}{\boldsymbol{U}}$*, where P denotes acoustic pressure and U denotes particle velocity*)

Also $v = \frac{1}{\sqrt{\mu\epsilon}}$ thus $Z = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu_r}{\epsilon_r}} Z_0$

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377\Omega$$

Skin Depth

Consider the plane wave solution

$$\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E_0 e^{i(k\cdot r-\omega t)}}$$

Substituting this solution in

$$\boldsymbol{\nabla^2 E} = \mu\sigma\frac{\partial \boldsymbol{E}}{\partial \mathrm{t}} + \mu\epsilon\frac{\partial^2 \mathbf{E}}{\partial \mathrm{t}^2}$$

$$-\mathrm{k}^2 = \mu\sigma(-i\omega) + \mu\epsilon(-\omega^2)$$

$$\mathrm{k}^2 = \mu\epsilon\omega^2\left(1 + \frac{i\sigma}{\omega\epsilon}\right)$$

Let $k = \alpha + i\beta$ thus $\mathrm{k}^2 = \alpha^2 - \beta^2 + 2\mathrm{i}\alpha\beta$

Comparing from above

$$\alpha^2 - \beta^2 = \mu\epsilon\omega^2$$

$$2\alpha\beta = \mu\omega\sigma$$

Solving for α, β we get

$$\alpha = \sqrt{\mu\epsilon}\omega\left(\frac{\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}+1}{2}\right)^{\frac{1}{2}}$$

$$\beta = \sqrt{\mu\epsilon}\omega\left(\frac{\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}-1}{2}\right)^{\frac{1}{2}}$$

$$\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E_0 e^{i\left((\alpha+i\beta)\hat{n}\cdot r-\omega t\right)}}$$

$$\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E_0 e^{-\beta\hat{n}\cdot r} e^{i(\alpha\hat{n}\cdot r-\omega t)}}$$

The field amplitudes are spatially attenuated: β is a measure of the attenuation, called the absorption coefficient or attenuation constant, while α is the phase constant (as $\sigma \to 0$, $\beta \to 0$ and $\alpha \to \omega\sqrt{\mu\epsilon}$, as expected). The fields propagate in the conducting medium with the phase speed

$$v = \frac{\omega}{\alpha} = \frac{1}{\sqrt{\mu\epsilon}}\left(\frac{\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}+1}{2}\right)^{-\frac{1}{2}}$$

For poor conductors / dielectrics

$$\frac{\sigma}{\omega\epsilon} \ll 1$$

$$v \approx \frac{1}{\sqrt{\mu\epsilon}}\left(1+\frac{1}{4}\left(\frac{\sigma}{\omega\epsilon}\right)^2\right)^{-\frac{1}{2}} \approx \frac{1}{\sqrt{\mu\epsilon}}$$

while the attenuation constant becomes

$$\beta \approx \sqrt{\mu\epsilon}\,\omega\left(\frac{1}{4}\left(\frac{\sigma}{\omega\epsilon}\right)^2\right)^{\frac{1}{2}} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}}$$

so that the skin depth of a poor conductor, $\delta = \frac{1}{\beta} \approx \frac{2}{\sigma}\sqrt{\frac{\epsilon}{\mu}}$, is independent of frequency: a slightly lossy dielectric carries the wave at almost the lossless speed with a small, frequency-independent attenuation.

For most dielectrics σ and ϵ vary with frequency in such a way that the ratio $\frac{\sigma}{\omega\epsilon}$ (the loss tangent) stays nearly constant over the frequency range of interest.

For good conductors

$$\frac{\sigma}{\omega\epsilon} \gg 1$$

$$\alpha \approx \beta = \sqrt{\frac{\sigma\omega\mu}{2}}$$

The phase speed inside a good conductor, $v = \frac{\omega}{\alpha} = \sqrt{\frac{2\omega}{\mu\sigma}}$, is tiny — about 400 m/s in copper at 1 MHz — and the wavelength inside the conductor is only $2\pi\delta$.

The distance $\frac{1}{\beta}$ over which the amplitude of a wave penetrating a conducting medium falls to 1/e of its value at the surface is called the skin depth or penetration depth:

$$\delta = \frac{1}{\beta} = \frac{1}{\sqrt{\mu\epsilon}\omega}\left(\frac{\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}-1}{2}\right)^{-\frac{1}{2}}$$

$$\delta = \frac{1}{\beta} = \sqrt{\frac{2}{\sigma\omega\mu}}$$

or, with $\omega = 2\pi f$,

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}}$$

Note that the skin depth decreases with increasing frequency. For copper ($\sigma = 5.8 \times 10^7$ S/m) it is about 9.3 mm at 50 Hz, 66 µm at 1 MHz and 2.1 µm at 1 GHz; for iron, with its large permeability, it is more than ten times smaller still.

Application: the induction cooker

An induction cooker is a transformer whose secondary is the base of the pan. A flat coil of many thin insulated copper strands beneath the glass-ceramic top carries an alternating current at 20–100 kHz; the changing magnetic field induces eddy currents in the pan, which heat it directly while the top stays comparatively cool. Like a step-down transformer, the pan sees a low voltage and a large current. Ferromagnetic cookware (iron, magnetic stainless steel) is required because its large permeability makes the skin depth very small — about 0.05 mm, against 0.5 mm in aluminium — so the induced current is confined to a thin surface layer of high resistance and produces strong Joule heating (hysteresis loss adds to it). Aluminium and copper pans, with their larger skin depth and lower resistance, heat poorly, which is why the cooker refuses to switch on for them. A lower frequency would penetrate deeper; shallow heating is what is wanted, since the heat is needed at the surface of the pan base.

Exercise:

1. $E = \frac{1V}{m}$, $H = \frac{0.01A}{m}$. Calculate Z. Ans: $Z = 100\Omega$
2. Co-axial cable: $Z = \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon}}\ln\left(\frac{b}{a}\right)$, given $a = 1mm, b = 5mm$, $\epsilon_r = 2.3, \mu_r = 1$. Find Z. Ans: $Z \approx 63.6\Omega$ — the characteristic impedance of the cable as a transmission line: the wave impedance of its dielectric, $\sqrt{\frac{\mu}{\epsilon}} = \frac{377}{\sqrt{2.3}} \approx 248\Omega$, multiplied by the geometrical factor $\frac{\ln(b/a)}{2\pi}$.

3. Find skin depth for copper given $\mu = 1.26 \times 10^{-6}\ H\ /m$
$\sigma = 5.8 \times 10^{7}\ S\ /m$
$\omega = 2 \times 10^{8}\ rad\ /s$
Ans: $\delta = 11.7\ \mu m$ (the given ω corresponds to f ≈ 32 MHz)

Applications

- Skin effect in power engineering: at 50 Hz the current in copper flows within about 9 mm of the surface, so very thick solid conductors are wasteful. High-voltage lines use aluminium strands around a steel core (ACSR) that carries the tension but little current, bus-bars are made hollow or flat, and radio-frequency coils use litz wire — many thin insulated strands, each thinner than the skin depth.
- Induction heating: the same shallow heating that cooks food hardens only the surface layer of gears and crankshafts (induction hardening) and melts metal in induction furnaces.
- Shielding and plating: a metal sheet a few skin depths thick stops an electromagnetic wave almost completely — at 1 GHz a few micrometres of copper suffice, which is why screening cans, the braid of a coaxial cable and the mesh in a microwave-oven door work. Waveguides and connectors are silver- or gold-plated only a few micrometres thick, because the current flows nowhere deeper.
- Communication through conductors: in sea water the skin depth is a few metres only at tens of hertz, so submarines are addressed by very-low-frequency transmitters; MRI scanners sit in a shielded room for the same reason, and the conductivity of the human body limits how deeply their radio-frequency pulses penetrate at high field strengths.
- Wave impedance: the 377 Ω of free space is what an antenna must be matched to in order to radiate efficiently, and the impedance ratio of two media fixes how much of a wave is reflected at their boundary — at normal incidence the reflected amplitude fraction is $(Z_2 - Z_1)/(Z_2 + Z_1)$ — the electromagnetic counterpart of the acoustic impedance mismatch that makes coupling gel necessary in ultrasound scanning.

Explore further

- Omni Calculator: Skin Depth Calculator — choose a metal and a frequency and read off δ; the page also works through copper at 50 Hz and at 2.4 GHz.
- Tatum, Physical Optics §2.5 “Impedance” (LibreTexts) — a compact derivation of Z = E/H = √(μ/ϵ) and the 377 Ω of free space, leading into reflection at boundaries.
- Falstad: Electrodynamics simulation (JavaScript, runs in the browser) — watch a plane wave meet a conducting block: the wave is reflected and penetrates only a thin layer.