Equation of Continuity and Boundary Conditions

If we apply the divergence to both sides of Ampère's law (with Maxwell's correction), $\boldsymbol{\nabla} \times \mathbf{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial \mathrm{t}}$, we obtain:

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{H}) = \boldsymbol{\nabla} \cdot \left(\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial \mathrm{t}} \right)$$

Since the divergence of a curl is identically zero, and $\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho$ (Gauss's law), this gives

$$\mathbf{0} = \boldsymbol{\nabla} \cdot \boldsymbol{J} + \frac{\partial \boldsymbol{\nabla} \cdot \boldsymbol{D}}{\partial \mathrm{t}}$$

$$\mathbf{0} = \boldsymbol{\nabla} \cdot \boldsymbol{J} + \frac{\partial \rho}{\partial \mathrm{t}}$$

$$\frac{\partial \rho}{\partial \mathrm{t}} = -\boldsymbol{\nabla} \cdot \boldsymbol{J}$$

The above is known as the equation of continuity. It expresses the conservation of charge: the charge inside any volume can change only by current flowing in or out through its surface. In general, an equation of continuity can be written for the flow of mass, energy, electric charge, momentum and even probability. The more general form of the equation of continuity is

$$\frac{\partial \rho}{\partial \mathrm{t}} + \boldsymbol{\nabla} \cdot \boldsymbol{J} = s$$

where, for some quantity q, $\rho = \frac{\mathrm{d}q}{\mathrm{dV}}$ is its density and $\boldsymbol{J} = \rho \boldsymbol{v}$ its flux (current) density, $\boldsymbol{v}$ being the flow velocity. Here s is the rate of generation of q per unit volume per unit time.

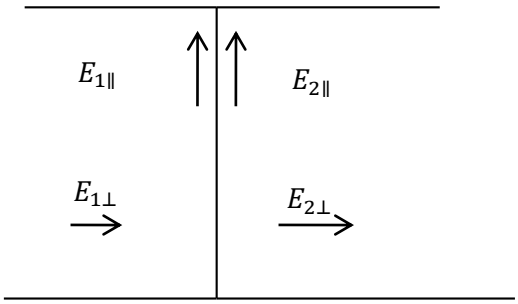
Regions where $s > 0$ or $s < 0$ are known as sources and sinks of q respectively.

If q is a conserved quantity, i.e. it cannot be created or destroyed, $s = 0$ (as for electric charge above).

Boundary conditions

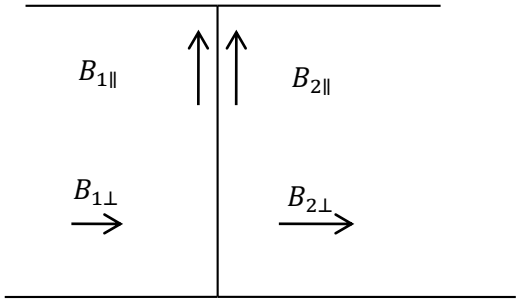
Boundary condition for the electric field

For an electric field crossing the interface between two media, the component parallel (tangential) to the interface is continuous, whereas the component perpendicular (normal) to the interface is discontinuous, the jump being set by the surface charge density on the interface.



Boundary condition for the magnetic field

For a magnetic field crossing the interface between two media, the component perpendicular (normal) to the interface is continuous, whereas the component parallel (tangential) to the interface is discontinuous, the jump being set by the surface current density on the interface.



Tangential component of the electric field is continuous:

$$E_{1\parallel} = E_{2\parallel}$$

Normal component of the electric field is discontinuous (the jump is set by the surface charge density):

$$E_{1\perp} - E_{2\perp} = \sigma/\epsilon_0$$

Normal component of the magnetic field is continuous:

$$B_{1\perp} = B_{2\perp}$$

Tangential component of the magnetic field is discontinuous (the jump is set by the surface current density):

$$B_{1\parallel} - B_{2\parallel} = \mu_0\kappa$$

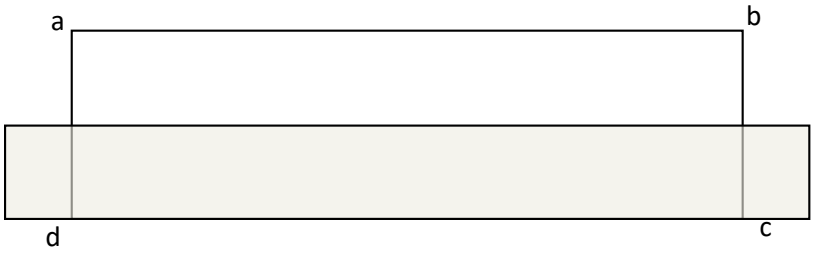
σ : surface charge density on the interface; κ : surface current density (current per unit length) on the interface.

In material media the same two conditions are written for the normal component of $\mathbf{D}$ and the tangential component of $\mathbf{H}$ in terms of the free surface charge and current densities: $D_{1\perp} - D_{2\perp} = \sigma_f$, $H_{1\parallel} - H_{2\parallel} = \kappa_f$.

Uses: finding the fields in waveguides, antennas and microwave circuits, and deriving the laws of reflection and refraction at an interface.

Derivation

Consider a small rectangular loop abcd straddling the interface between medium 1 and medium 2 (figure).



The shaded region is medium 2; if it is a conductor, the fields inside it are zero. Applying Faraday's law to the loop abcd:

$$\int_a^b \boldsymbol{E}\cdot d\boldsymbol{l} + \int_b^c \boldsymbol{E}\cdot d\boldsymbol{l} + \int_c^d \boldsymbol{E}\cdot d\boldsymbol{l} + \int_d^a \boldsymbol{E}\cdot d\boldsymbol{l} = -\frac{d}{dt}\int_{abcd} \boldsymbol{B}\cdot d\boldsymbol{S}$$

As the short sides *ad* and *bc* are made vanishingly small, the integrals along them vanish:

$$\int_a^b \boldsymbol{E}\cdot d\boldsymbol{l} + \int_c^d \boldsymbol{E}\cdot d\boldsymbol{l} = -\frac{d}{dt}\int_{abcd} \boldsymbol{B}\cdot d\boldsymbol{S}$$

Since the area enclosed by the loop also tends to zero, the magnetic flux through it vanishes, and the above reduces to

$$\int_a^b \boldsymbol{E}\cdot d\boldsymbol{l} + \int_c^d \boldsymbol{E}\cdot d\boldsymbol{l} = 0$$

$$E_{1\parallel}L + E_{2\parallel}(-L) = 0$$

$$E_{1\parallel} = E_{2\parallel}$$

Thus the tangential component of the electric field is continuous across the interface. If medium 2 is a conductor, $E_{2\parallel} = 0$ and hence $E_{1\parallel} = 0$: the tangential electric field vanishes at the surface of a conductor.

Derivation of the remaining conditions

Normal component of $\mathbf{E}$: apply Gauss's law to a small flat cylinder (a "pillbox") of face area A straddling the interface, with its faces parallel to the interface and its height shrunk to zero, so that the curved side contributes nothing to the flux. The enclosed charge is the surface charge σA:

$$\oint \boldsymbol{E} \cdot \boldsymbol{dS} = \frac{q_{enclosed}}{\epsilon_0}$$

$$E_{1\perp}A - E_{2\perp}A = \frac{\sigma A}{\epsilon_0}$$

giving $E_{1\perp} - E_{2\perp} = \sigma/\epsilon_0$. The normal component of $\mathbf{E}$ jumps by σ/ϵ_0; at an interface carrying no surface charge it is continuous.

Normal component of $\mathbf{B}$: the same pillbox with Gauss's law for magnetism,

$$\oint \boldsymbol{B} \cdot \boldsymbol{dS} = 0$$

$$B_{1\perp}A - B_{2\perp}A = 0$$

gives $B_{1\perp} = B_{2\perp}$: the normal component of $\mathbf{B}$ is always continuous, because there are no magnetic charges to produce a jump.

Tangential component of $\mathbf{B}$: apply Ampère's law to the rectangular loop abcd of the figure. As the short sides shrink to zero the displacement-current term $\frac{\partial \mathbf{D}}{\partial \mathrm{t}}$ vanishes together with the loop area, so only the conduction current threading the loop survives. If a surface current of density κ (current per unit length) flows on the interface at right angles to the plane of the loop, the current threading the loop is κL:

$$\oint \boldsymbol{B} \cdot \boldsymbol{dl} = \mu_0 I_{enclosed}$$

$$B_{1\parallel}L - B_{2\parallel}L = \mu_0 \kappa L$$

giving $B_{1\parallel} - B_{2\parallel} = \mu_0 \kappa$. Across an interface carrying no surface current ($\kappa = 0$), as between two dielectrics, the tangential component of $\mathbf{B}$ is continuous as well.

Applications

- Steady currents: with $\frac{\partial \rho}{\partial \mathrm{t}} = 0$ the continuity equation gives $\boldsymbol{\nabla} \cdot \mathbf{J} = 0$ — the current flowing into any junction equals the current flowing out. This is Kirchhoff's current law.
- Charge relaxation in a conductor: combining $\frac{\partial \rho}{\partial \mathrm{t}} = -\boldsymbol{\nabla} \cdot \mathbf{J}$ with $\mathbf{J} = \sigma \mathbf{E}$ and $\boldsymbol{\nabla} \cdot \mathbf{E} = \rho/\epsilon$ gives $\frac{\partial \rho}{\partial \mathrm{t}} = -\frac{\sigma}{\epsilon}\rho$, so any charge placed inside a conductor decays as $e^{-\sigma t/\epsilon}$ with time

constant $\epsilon/\sigma \approx 10^{-19}$ s for copper — excess charge can reside only on the surface of a conductor.

- Semiconductor devices: the continuity equations for electrons and holes, with generation and recombination as the source term s, are the equations solved to design every diode, transistor and solar cell.
- The same equation expresses conservation of mass in fluid flow, of energy in heat conduction, and of probability in quantum mechanics.
- $E_{\parallel} = 0$ at a conductor surface is why the field inside a closed metal enclosure is zero (the Faraday cage): the shielding of coaxial cables, of sensitive electronics, and of the occupants of a car struck by lightning.
- Continuity of the tangential field and of the phase of the wave along a dielectric interface is where the laws of reflection and refraction (Snell's law) come from; the conditions on $E_{\parallel}$ and $B_{\perp}$ at metal walls decide which modes a waveguide can carry, and the metal mesh in a microwave-oven door is a boundary condition in action.

Explore further

- The Feynman Lectures on Physics, Vol. II, §13-2 "Electric current; the conservation of charge" (free online) — the clearest short treatment of $\boldsymbol{\nabla} \cdot \mathbf{J} = -\frac{\partial \rho}{\partial \mathrm{t}}$ and of why charge cannot simply disappear.
- Falstad: Electrodynamics simulation (JavaScript, runs in the browser) — its conductor set-ups show the tangential electric field vanishing at a metal surface and the wave reflecting, i.e. the boundary condition above seen live.