Electromagnetic wave equation

Consider a uniform, source-free medium (no free charge, $\rho = 0$, and no impressed current sources) having permittivity ϵ, magnetic permeability μ, and conductivity σ

Uniform medium

Linear : $\mathbf{D}$ is proportional to **E** ($\mathbf{D} = \epsilon\mathbf{E}$)

: $\mathbf{B}$ is proportional to **H** ($\mathbf{B} = \mu\mathbf{H}$)

Homogeneous : Medium properties are same at all points

Isotropic : μ and ϵ are independent of direction (μ and ϵ are scalar constants), so $\mathbf{D}$ is parallel to **E** and $\mathbf{B}$ is parallel to **H**.

Maxwell's equations

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\mu\frac{\partial \mathbf{H}}{\partial t}$$

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \sigma\boldsymbol{E} + \epsilon\frac{\partial \mathbf{E}}{\partial t}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{H} = 0$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0$$

To obtain an equation in **E** alone, take the curl of both sides of the first equation:

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{E}) = -\mu\boldsymbol{\nabla} \times \left(\frac{\partial \mathbf{H}}{\partial t}\right)$$

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$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{E}) = \left(-\mu\frac{\partial}{\partial t}\left(\sigma\boldsymbol{E} + \epsilon\frac{\partial \mathbf{E}}{\partial t}\right)\right)$$

Identity

$$\mathbf{A} \times (\mathbf{B} \times \boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{A} \cdot \boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A} \cdot \boldsymbol{B})$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{E}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{E}) - (\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\boldsymbol{E} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{E}) - \boldsymbol{\nabla^2}\boldsymbol{E}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{E}) = -\boldsymbol{\nabla^2}\boldsymbol{E}$$

since $\boldsymbol{\nabla \cdot E} = 0$ in a source-free medium.

$$-\boldsymbol{\nabla^2 E} = -\mu\sigma\frac{\partial \boldsymbol{E}}{\partial \mathrm{t}} - \mu\epsilon\frac{\partial^2 \mathbf{E}}{\partial \mathrm{t}^2}$$

$$\boldsymbol{\nabla^2 E} = \mu\sigma\frac{\partial \boldsymbol{E}}{\partial \mathrm{t}} + \mu\epsilon\frac{\partial^2 \mathbf{E}}{\partial \mathrm{t}^2}$$

Similarly

$$\boldsymbol{\nabla^2 H} = \mu\sigma\frac{\partial \boldsymbol{H}}{\partial \mathrm{t}} + \mu\epsilon\frac{\partial^2 \mathbf{H}}{\partial \mathrm{t}^2}$$

These are known as the general wave equations.

These equations govern the behavior of an electromagnetic field in a uniform but source-free conducting medium. The presence of the first-order term $\frac{\partial}{\partial \mathrm{t}}$ indicates that the fields decay as they propagate through the medium (energy is dissipated as Joule heat by the conduction current). For this reason, a conducting medium is called a lossy medium.

Plane wave in a dielectric medium

Here $\sigma = 0$ (no conduction current)

$$\boldsymbol{\nabla^2 E} - \mu\epsilon\frac{\partial^2 \mathbf{E}}{\partial \mathrm{t}^2} = 0$$

$$\boldsymbol{\nabla^2 H} - \mu\epsilon\frac{\partial^2 \mathbf{H}}{\partial \mathrm{t}^2} = 0$$

These are the homogeneous vector wave equations. For time-harmonic fields varying as $e^{j\omega t}$ they reduce to the Helmholtz equations $\boldsymbol{\nabla^2 E} + k^2\boldsymbol{E} = 0$, with $k = \omega\sqrt{\mu\epsilon}$.

The absence of the first-order term indicates that electromagnetic fields do not decay as they propagate in a lossless medium.

Speed of electromagnetic waves

Comparing the dielectric-medium equations with the standard wave equation

$$\boldsymbol{\nabla^2} u - \frac{1}{v^2}\frac{\partial^2 u}{\partial \mathrm{t}^2} = 0$$

shows that the fields propagate as waves with speed

$$v = \frac{1}{\sqrt{\mu\epsilon}}$$

In free space ($\mu = \mu_0 = 4\pi \times 10^{-7}$ H/m, $\epsilon = \epsilon_0 = 8.854 \times 10^{-12}$ F/m)

$$v = c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3 \times 10^8 \text{ m/s}$$

This is Maxwell's identification of light as an electromagnetic wave: the speed of light follows from two constants measured in the laboratory with charges, coils and capacitors. In a material medium the wave travels more slowly, v = c/n, where the refractive index is

$$n = \frac{c}{v} = \sqrt{\mu_r \epsilon_r} \approx \sqrt{\epsilon_r}$$

the approximation holding for non-magnetic media ($\mu_r \approx 1$). Here ϵ_r is the relative permittivity at the frequency of the wave, not the static value: for water the static value is about 81 but at optical frequencies n = 1.33, because the slow orientation of the water molecules cannot follow an optical-frequency field.

The electromagnetic spectrum

All electromagnetic waves travel at c in vacuum and differ only in frequency (wavelength $\lambda = c/f$); they are generated by different mechanisms and interact with matter differently. The visible band is a very narrow slice of the whole.

Region	Wavelength	Frequency (Hz)	Typical sources and uses
Radio waves	> 1 m	< 3×10^8	AM/FM and TV broadcasting, MRI (radio-frequency pulses), radio astronomy
Microwaves	1 mm – 1 m	3×10^8 – 3×10^{11}	mobile phones, Wi-Fi, radar, satellite links, microwave ovens (2.45 GHz)
Infrared	700 nm – 1 mm	3×10^{11} – 4.3×10^{14}	thermal imaging, remote controls, optical-fibre communication (1.3–1.55 µm)
Visible	400 – 700 nm	4.3×10^{14} – 7.5×10^{14}	vision, lasers, LEDs, photography, spectroscopy
Ultraviolet	10 – 400 nm	7.5×10^{14} – 3×10^{16}	sterilization, fluorescence, photolithography, sunburn
X-rays	0.01 – 10 nm	3×10^{16} – 3×10^{19}	medical radiography and CT, crystallography, security scanners
Gamma rays	< 0.01 nm	> 3×10^{19}	nuclear decay, cancer radiotherapy, PET imaging, astrophysics

The boundaries between regions are conventional and overlap. The photon energy $E = hf$ rises from about 10^{-6} eV for radio waves to above 10^5 eV for gamma rays, which is why X-rays and gamma rays ionize atoms (and damage tissue) while radio waves and microwaves do not.

Applications

- The loss term $\mu\sigma\frac{\partial \mathbf{E}}{\partial t}$: sea water is a conductor, so ordinary radio waves are absorbed within metres of the surface — submarines are contacted with very-low-frequency waves, which

penetrate deepest. The same term is why mobile signal fades inside steel-framed buildings and lifts, and why the metal mesh in a microwave-oven door stops the 2.45 GHz waves.

- In a lossless dielectric the waves travel undiminished — the principle behind glass optical fibres carrying signals over tens of kilometres, and behind radio reaching us through the (nearly lossless) air.
- $v = 1/\sqrt{\mu\epsilon}$ is the origin of the refractive index and of the speed of light; the finite value of c sets the ~0.25 s round-trip delay of a geostationary satellite link and limits how fast signals can cross a computer chip.
- The spectrum table is a map of modern technology: every band from radio to gamma rays is the same wave equation at a different frequency.

Explore further

- PhET: Radio Waves & Electromagnetic Fields (free, runs in the browser) — wiggle the electron in a transmitting antenna and watch the electric-field wave travel to a receiver, as a curve or as field vectors.
- Falstad: Electrodynamics simulation (JavaScript, runs in the browser) — two-dimensional electromagnetic waves: plane waves, reflection from conductors, antennas, standing waves and waveguides.