Vector Analysis: Gradient, Divergence and Curl

Cartesian Coordinates

Gradient of any scalar function like $\varphi(x,y,z)$ can be any scalar function like $\varphi(x,y,z) = x^2 - xy + z$

$$\boldsymbol{\nabla}\varphi(x,y,z) = \frac{\partial\varphi}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial\varphi}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial\varphi}{\partial z}\hat{\boldsymbol{k}}$$

Gradient points in the direction of steepest increase of ϕ; its magnitude is that maximum rate of change.

Electric field E is a gradient of electrostatic potential V

$$\boldsymbol{E} = -\boldsymbol{\nabla}V$$

Only for symmetric cases of V — e.g. a point charge — can we write

$\boldsymbol{E} = -\frac{\partial V}{\partial r}\hat{\boldsymbol{r}}$ or for a capacitor we can write $\boldsymbol{E} = \frac{V}{d}$

Divergence (Rate at which a vector field diverges from a point) of $\boldsymbol{A} = A_x\hat{\boldsymbol{\imath}} + A_y\hat{\boldsymbol{\jmath}} + A_z\hat{\boldsymbol{k}}$: $\boldsymbol{A}$ can be any vector with components A_x, A_y, A_z as functions of x , y and z. $\boldsymbol{A} = (x^2 - xy + z)\hat{\boldsymbol{\imath}} + (x^3 - xz + x)\hat{\boldsymbol{\jmath}} + (y^2 - y + z)\hat{\boldsymbol{k}}$

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Curl(Circulation of vector field around a point)

Curl of $\boldsymbol{A} = A_x\hat{\boldsymbol{\imath}} + A_y\hat{\boldsymbol{\jmath}} + A_z\hat{\boldsymbol{k}}$

$$\boldsymbol{\nabla}\times\boldsymbol{A} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Cylindrical coordinates (ρ, ϕ, z)

$x = \rho\cos\phi\ y = \rho\sin\phi\ z = z$

$$\boldsymbol{\nabla}\varphi(\rho,\phi,z) = \frac{\partial\varphi}{\partial\rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial\varphi}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial\varphi}{\partial z}\hat{\boldsymbol{z}}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{A}=\frac{1}{\rho}\left[\frac{\partial\left(\rho A_{\rho}\right)}{\partial\rho}+\frac{\partial A_{\phi}}{\partial\phi}+\frac{\partial\left(\rho A_{z}\right)}{\partial z}\right]$$

$$\boldsymbol{\nabla}\times\boldsymbol{A}=\frac{1}{\rho}\begin{vmatrix}\hat{\rho} & \rho\hat{\phi} & \hat{z}\\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z}\\ A_{\rho} & \rho A_{\phi} & A_{z}\end{vmatrix}$$

Spherical coordinates (r,θ,ϕ)

$$x=r\sin\theta\cos\phi\ y=r\sin\theta\sin\phi\ z=r\cos\theta$$

$$\boldsymbol{\nabla}\varphi(r,\theta,\phi)=\frac{\partial\varphi}{\partial r}\hat{r}+\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\hat{\theta}+\frac{1}{r\sin\theta}\frac{\partial\varphi}{\partial\phi}\hat{\phi}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{A}=\frac{1}{r^{2}\sin\theta}\left[\frac{\partial(r^{2}\sin\theta\,A_{r})}{\partial r}+\frac{\partial(r\sin\theta\,A_{\theta})}{\partial\theta}+\frac{\partial\left(rA_{\phi}\right)}{\partial\phi}\right]$$

$$\boldsymbol{\nabla}\times\boldsymbol{A}=\frac{1}{r^{2}\sin\theta}\begin{vmatrix}\hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\phi}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi}\\ A_{r} & rA_{\theta} & r\sin\theta\,A_{\phi}\end{vmatrix}$$

Website

https://falstad.com/vector/

https://www.3blue1brown.com/lessons/divergence-and-curl/

Applications