Wave impedance

General wave equation

$$\nabla^2 \boldsymbol{u} - \frac{1}{v^2}\frac{\partial^2 \mathbf{u}}{\partial \mathrm{t}^2} = 0$$

$$v = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{\sqrt{\mu_r \epsilon_r}}$$

The plane wave solutions are

$$\boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{E_0} \boldsymbol{e}^{\boldsymbol{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\boldsymbol{\omega t})}$$

$$\boldsymbol{H}(\boldsymbol{r}, t) = \boldsymbol{H_0} \boldsymbol{e}^{\boldsymbol{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\boldsymbol{\omega t})}$$

The following Maxwell equations give

$$\nabla \times \boldsymbol{E} = -\mu \frac{\partial \mathbf{H}}{\partial \mathrm{t}} \qquad \nabla \times \boldsymbol{H} = \epsilon \frac{\partial \mathbf{E}}{\partial \mathrm{t}}$$

$$\mathbf{k} \times \boldsymbol{E} = \mu\omega\mathbf{H} \qquad \mathbf{k} \times \boldsymbol{H} = -\epsilon\omega\mathbf{E}$$

The third equation gives

$$\frac{\omega\mu}{\mathrm{k}} = \left|\frac{E}{\mathrm{H}}\right|$$

Since $v = \frac{\omega}{k}$

Wave impedance $\quad Z = \left|\frac{\boldsymbol{E}}{\boldsymbol{H}}\right| = v\mu$

(*In Acoustics the expression of wave impedance is given by* $Z = \frac{\boldsymbol{P}}{\boldsymbol{U}}$*, where P denotes acoustics pressure and U denotes particle velocity*)

Also $= \frac{1}{\sqrt{\mu\epsilon}}$ thus $Z = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu_r}{\epsilon_r}} Z_0$

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377\Omega$$

Version 4.0

Skin Depth

Consider the plane wave solution

$$\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E_0} e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\boldsymbol{\omega t})}$$

Substituting this solution in

$$\nabla^2 \boldsymbol{E} = \mu\sigma\frac{\partial \boldsymbol{E}}{\partial \mathrm{t}} + \mu\epsilon\frac{\partial^2 \mathbf{E}}{\partial \mathrm{t}^2}$$

$$-\mathrm{k}^2 = \mu\sigma(-i\omega) + \mu\epsilon(-\omega^2)$$

$$\mathrm{k}^2 = \mu\epsilon\omega^2\left(1 + \frac{i\sigma}{\omega\epsilon}\right)$$

Let $k = \alpha + i\beta$ thus $\mathrm{k}^2 = \alpha^2 - \beta^2 + 2\mathrm{i}\alpha\beta$

Comparing from above

$$\alpha^2 - \beta^2 = \mu\epsilon\omega^2$$

$$2\alpha\beta = \mu\omega\sigma$$

Solving for α, β we get

$$\alpha = \sqrt{\mu\epsilon}\omega\left(\frac{\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2} - 1}{2}\right)^{\frac{1}{2}}$$

$$\beta = \sqrt{\mu\epsilon}\omega\left(\frac{\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2} + 1}{2}\right)^{\frac{1}{2}}$$

$$\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E_0} e^{i\left((\alpha+i\beta)\hat{\boldsymbol{n}}\cdot\boldsymbol{r}-\boldsymbol{\omega t}\right)}$$

$$\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E_0} e^{-\beta\hat{\boldsymbol{n}}\cdot\boldsymbol{r}} e^{i(\alpha\hat{\boldsymbol{n}}\cdot\boldsymbol{r}-\omega t)}$$

Field amplitudes are spatially attenuated β is a measure of attenuation called absorption coefficient or attenuation constant. Also vector field are propagated in the conducting medium with a speed

$$v = \frac{\omega}{\alpha} = \frac{1}{\sqrt{\mu\epsilon}}\left(\frac{\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}-1}{2}\right)^{-\frac{1}{2}}$$

For poor conductors/ Dielectrics

$$\frac{\sigma}{\omega\epsilon} \ll 1$$

$$v = \frac{1}{\sqrt{\mu\epsilon}}\left(\frac{1+\frac{1}{2}\left(\frac{\sigma}{\omega\epsilon}\right)^2-1}{2}\right)^{-\frac{1}{2}} = \frac{2\omega}{\sigma}\sqrt{\frac{\epsilon}{\mu}}$$

For most dielectrics σ and ϵ are a function of frequency while the term $\frac{\sigma}{\omega\epsilon}$ is constant over the frequency range of interest.

For Good conductors

$$\frac{\sigma}{\omega\epsilon} \gg 1$$

$$\alpha \approx \beta = \sqrt{\frac{\sigma\omega\mu}{2}}$$

The term $\frac{1}{\beta}$, measures the depth at which the electro magnetic wave penetrating a conducting medium is damped to 1/e of its initial amplitude at its surface. The depth is called penetration depth.

$$\delta = \frac{1}{\beta} = \frac{1}{\sqrt{\mu\epsilon}\omega}\left(\frac{\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}+1}{2}\right)^{-\frac{1}{2}}$$

$$\delta = \frac{1}{\beta} = \sqrt{\frac{2}{\sigma\omega\mu}}$$

Note that skin depth decreases with increase in frequency

Application

Induction cooker

Induction cooker requires ferromagnetic cookware so that the skin depth in the cookware is small and thus large current. The induction cooker works as a step down transformer. The voltage is reduced, the current increases. A thick bunch of copper wires is used beneath the glass ceramic surface. Lower frequency leads to deeper penetration. Shallow heating is desirable as head needs to go to surface.

Exercise:

1. $E = \frac{1V}{m}, H = \frac{0.01A}{m}$. Calculate Z. Ans: $Z = 100\Omega$

2.Co-axial cable: $Z = \frac{1}{\sqrt{\mu\epsilon}} \ln\left(\frac{b}{a}\right)$, given $a = 1mm, b = 5mm$, $\epsilon_r = 2.3, \mu_r = 1$. Find Z.

Ans: $Z = 1.061\Omega$

2. Find skin depth for copper given $\mu = 1.26 \times 10^{-6}\ H/m$

$$\sigma = 5.8 \times 10^{7}\ S/m$$
$$\omega = 2 \times 10^{8}\ rad/s$$

   Ans: $\delta = 2.82mm$