Equation of Continuity

If we apply the divergence to both sides of Ampere's Law, then we obtain:

$$\nabla \cdot \left(\nabla \times \mathbf{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial \mathrm{t}}\right)$$

$$\nabla \cdot (\nabla \times \mathbf{H}) = \nabla \cdot \left(\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial \mathrm{t}}\right)$$

$$\mathbf{0} = \nabla \cdot \boldsymbol{J} + \frac{\partial \nabla \cdot \boldsymbol{D}}{\partial \mathrm{t}}$$

$$\mathbf{0} = \nabla \cdot \boldsymbol{J} + \frac{\partial \rho}{\partial \mathrm{t}}$$

$$\frac{\partial \rho}{\partial \mathrm{t}} = -\nabla \cdot \boldsymbol{J}$$

The above is known as equation of continuity. In general equation of continuity can be applied to flow of mass, energy, electric charge, momentum and even probability. The more general form of equation of continuity is

$$\frac{\partial \rho}{\partial \mathrm{t}} + \nabla \cdot \boldsymbol{J} = \sigma$$

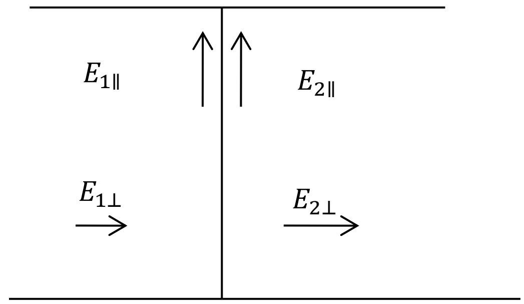
Where for some quantity$q$, $\rho = \frac{\mathrm{d}q}{\mathrm{dV}}$ and $\boldsymbol{J} = \rho \boldsymbol{v}$. Here$\sigma$ is generation of q per unit volume per unit time.

Terms that generate $\sigma > 0$ or $\sigma < 0$ are known as sources and sinks of q respectively.

In case q is a conserved quantity, i.e. cannot be created or destroyed $\sigma = 0$.

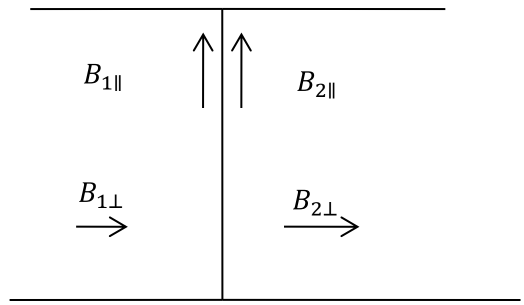
Boundary conditions

Boundary condition for the electric field



For an electric field crossing the boundary of two surfaces, the component parallel to the surface is continuous, however, the component perpendicular the surface has discontinuity and is related to the surface charge density of that interface.

Boundary condition for the Magnetic field



For a Magnetic field crossing the boundary of two surfaces, the component perpendicular to the surface is continuous, however, the component parallel to the surface has discontinuity and is related to the surface current density of the interface.

Tangential Component of Electric Field is continuous

$$E_{1\parallel} = E_{2\parallel}$$

$$E_{1\perp} - E_{2\perp} = \sigma/\epsilon_0$$

$$B_{1\parallel} - B_{2\parallel} = \kappa$$

Perpendicular Component of Magnetic Field is continuous

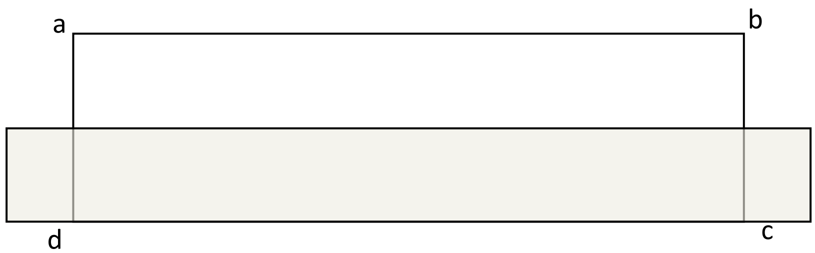
$$B_{1\perp} = B_{2\perp}$$

κ : surface current density

Uses: waveguides, antennas, microwave circuits.

Derivation

Consider the surface



Field inside the conductor are zero.

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$$\int_a^b \boldsymbol{E}\cdot d\boldsymbol{l} + \int_b^c \boldsymbol{E}\cdot d\boldsymbol{l} + \int_c^d \boldsymbol{E}\cdot d\boldsymbol{l} + \int_d^a \boldsymbol{E}\cdot d\boldsymbol{l} = -\frac{d}{dt}\int_{abcd} \boldsymbol{B}\cdot d\boldsymbol{S}$$

Limiting *ad* and *bc* to be very small

$$\int_a^b \boldsymbol{E}\cdot d\boldsymbol{l} + \int_b^c \boldsymbol{E}\cdot d\boldsymbol{l} + \int_c^d \boldsymbol{E}\cdot d\boldsymbol{l} + \int_d^a \boldsymbol{E}\cdot d\boldsymbol{l} = -\frac{d}{dt}\int_{abcd} \boldsymbol{B}\cdot d\boldsymbol{S}$$

Since area enclosed is very small, also ad and bc are very small, thus the above integral reduces to

$$\int_a^b \boldsymbol{E}\cdot d\boldsymbol{l} + \int_c^d \boldsymbol{E}\cdot d\boldsymbol{l} = 0$$

$$E_{1\parallel}L + E_{2\parallel}(-L) = 0$$

$$E_{1\parallel} = E_{2\parallel}$$