Vector Analysis: Gradient, Divergence and Curl

Cartesian Coordinates

Gradient of (x,y,z) : $\varphi(x,y,z)$ can be any scalar function like $\varphi(x,y,z) = x^2 - xy + z$

$$\nabla\varphi(x,y,z) = \frac{\partial\varphi}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial\varphi}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial\varphi}{\partial z}\hat{\boldsymbol{k}}$$

Electric field E is a gradient of electrostatic potential V

$$\boldsymbol{E} = -\nabla V$$

It is only for symmetrical cases of V like for a point charge we can write $\boldsymbol{E} = -\frac{\partial V}{\partial r}\hat{\boldsymbol{r}}$ or for a capacitor we can write $\boldsymbol{E} = \frac{V}{d}$

Divergence (Rate at which a vector field diverges from a point) of $\boldsymbol{A} = A_x\hat{\boldsymbol{\imath}} + A_y\hat{\boldsymbol{\jmath}} + A_z\hat{\boldsymbol{k}}$: $\boldsymbol{A}$ can be any vector with coefficients A_x, A_y, A_z as functions of x , y and z. $\boldsymbol{A} = (x^2 - xy + z)\hat{\boldsymbol{\imath}} + (x^3 - xz + x)\hat{\boldsymbol{\jmath}} + (y^2 - y + z)\hat{\boldsymbol{k}}$

$$\nabla\cdot\boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Curl(Circulation of vector field around a point)

Curl of $\boldsymbol{A} = A_x\hat{\boldsymbol{\imath}} + A_y\hat{\boldsymbol{\jmath}} + A_z\hat{\boldsymbol{k}}$

$$\nabla\times\boldsymbol{A} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Cylindrical coordinates (ρ, ϕ, z)

$x = \rho\cos\phi$ $y = \rho\sin\phi$ $z = z$

$$\nabla\varphi(\rho,\phi,z) = \frac{\partial\varphi}{\partial\rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial\varphi}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial\varphi}{\partial z}\hat{\boldsymbol{z}}$$

$$\nabla\cdot\boldsymbol{A} = \frac{1}{\rho}\left[\frac{\partial(\rho A_\rho)}{\partial\rho} + \frac{\partial A_\phi}{\partial\phi} + \frac{\partial(\rho A_z)}{\partial z}\right]$$

$$\nabla \times \boldsymbol{A} = \frac{1}{\rho} \begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & A_\phi & A_z \end{vmatrix}$$

Spherical coordinates (r, θ, ϕ)

$$x = r\sin\theta\cos\phi \; y = r\sin\theta\sin\phi \; z = r\cos\theta$$

$$\nabla\varphi(\rho, \theta, \phi) = \frac{\partial \varphi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial \varphi}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial \varphi}{\partial \phi}\hat{\phi}$$

$$\nabla \cdot \boldsymbol{A} = \frac{1}{r^2\sin\theta}\left[\frac{\partial(r^2\sin\theta\, A_r)}{\partial r} + \frac{\partial(r\sin\theta\, A_\theta)}{\partial \theta} + \frac{\partial\left(rA_\phi\right)}{\partial \phi}\right]$$

$$\nabla \times \boldsymbol{A} = \frac{1}{r^2\sin\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta\, A_\phi \end{vmatrix}$$